

# Smooth Rényi Entropies and the Quantum Information Spectrum

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Many of the traditional results in information theory, such as the channel coding theorem or the source coding theorem, are restricted to scenarios where the underlying resources are *independent and identically distributed (i.i.d.)* over a large number of uses. To overcome this limitation, two different techniques, the *information spectrum method* and the *smooth entropy framework*, have been developed independently. They are based on new entropy measures, called *spectral entropy rates* and *smooth entropies*, respectively, that generalize Shannon entropy (in the classical case) and von Neumann entropy (in the more general quantum case). Here, we show that the two techniques are closely related. More precisely, the spectral entropy rate can be seen as the asymptotic limit of the smooth entropy. Our results apply to the quantum setting and thus include the classical setting as a special case.

## INTRODUCTION

Traditional results in information theory, e.g., the *noisy channel coding theorem* or the *source coding (or data compression) theorem*, typically rely on the assumption that underlying resources, e.g., information sources and communication channels, are “memoryless”. A memoryless information source is one which emits signals that are independent of each other. Similarly, a channel is said to be memoryless if the noise acting on successive inputs to the channel is uncorrelated. Such resources can be described by a sequence of *identical and independently distributed (i.i.d.)* random variables.

In reality, however, this assumption cannot generally be justified. This is particularly problematic in cryptography, where the accurate modeling of the system is essential to derive any claim about its security.

In the past decade, two approaches have been proposed independently to overcome this limitation. The *information spectrum approach* was introduced by Han and Verdú [11, 12, 28] in an attempt to generalize the noisy channel coding theorem. This approach yields a unifying mathematical framework for obtaining asymptotic rate formulae for many different operational schemes in information theory, such as data compression, data transmission, and hypothesis testing. The power of this method lies in the fact that it does not rely on the specific nature of the sources or channels involved in the schemes.

The main ingredients of this method are new entropy-type measures, called *spectral entropy rates*, which are defined asymptotically for sequences of probability distributions. They can be seen as generalizations of the Shannon entropy, and also inherit many of its properties, such as subadditivity, strong subadditivity, monotonicity, and Araki-Lieb inequalities. They also satisfy chain rule inequalities. Their main feature, however, is that they characterize various other asymptotic information-theoretic quantities, e.g., the *data compression rate*, without relying on the i.i.d. assumption.

Subsequently, Hayashi, Nagaoka, and Ogawa have generalized the information-spectrum method to quantum-mechanical settings. They have applied the method to study quantum hypothesis testing and quantum source coding [16, 18], as well as to determine general expressions for the optimal rate of entanglement concentration [14] and the classical capacity of quantum channels [13]. The method has been further extended by Bowen and Datta [3] and used to obtain general formulae for the optimal rates of various information-theoretic protocols, e.g., the dense coding capacity for a noiseless quantum channel, assisted by arbitrary shared entanglement [4] and the entanglement cost for arbitrary sequences of pure [5] and mixed [6] states. Recently, Matsumoto [15] has also employed the information spectrum method to obtain an alternative (but equivalent) expression for the entanglement cost for an arbitrary sequence of states.

In a simultaneous but independent development, the necessity to generalize Shannon’s theory became apparent in the context of *cryptography*. Roughly speaking, one of the main challenges in cryptography is that one needs to deal with an adversary who might pursue an *arbitrary* (and unknown) strategy. In particular, the adversary might introduce undesired correlations which, for instance, make it difficult to justify assumptions on the independence of noise in a communication channel.

Bennett, Brassard, Crépeau, and Maurer [1] were among the first to make this point explicit, arguing that the Shannon entropy is not an appropriate measure for the ignorance of an adversary about a (partially secret) key. They proposed an alternative measure based on the collision entropy (i.e., Rényi entropy [24] of order 2) and a notion called *spoiling knowledge*, which can be seen as a predecessor of smooth entropies. This approach has been further investigated by Cachin [7], who also found connections to other entropy measures, in particular *Rényi entropies* of arbitrary order.

Motivated by the work of Bennett *et al.* and Cachin, *smooth Rényi entropies* have been introduced by Ren-

ner et al., first for the purely classical case (in [22]), and later for the more general quantum regime (in [20, 21]). In contrast to the spectral entropy rates, smooth Rényi entropies are defined for single distributions (rather than sequences of distributions). Because of their *non-asymptotic* nature, they depend on an additional parameter  $\varepsilon$ , called *smoothness*.

Similarly to the spectral entropy rates, it has been shown that smooth entropies have many properties in common with Shannon and von Neumann entropy (for example, there is a chain rule, and strong subadditivity holds) [20, 23]. Furthermore, they allow for a quantitative analysis of a broad variety of information-theoretic tasks—but in contrast to Shannon entropy, neither the i.i.d. assumption nor asymptotics are needed. For example, in the classical regime, it is possible to give a fully general formula for the number of classical bits that can be transmitted reliably (up to some error  $\varepsilon$ ) in *one* (or finitely many) uses of a classical channel [25]. In the quantum regime, they proved very useful in the context of randomness extraction [20, 21], which, in turn, is used for cryptographic applications [8, 9, 10, 27]. In particular, they are employed for the study of real-world implementations of cryptographic schemes, where the available resources (e.g., the computational power or the memory size) are finite [26].

Our aim in this paper is to find connections between the two different approaches described above, by exploring the relationships between spectral entropy rates and smooth entropies. We do this in two steps. First, we consider the special case where the entropies are not conditioned on an additional system, in the following called the *non-conditional case*. Then, in a second step, we consider the general *conditional case* where the entropies are conditioned on an extra system.

## DEFINITIONS OF SMOOTH ENTROPY AND SPECTRAL ENTROPY RATES

### Mathematical Preliminaries

Let  $\mathcal{B}(\mathcal{H})$  denote the algebra of linear operators acting on a finite-dimensional Hilbert space  $\mathcal{H}$ . The von Neumann entropy of a state  $\rho$ , i.e., a positive operator of unit trace in  $\mathcal{B}(\mathcal{H})$ , is given by  $S(\rho) = -\text{Tr} \rho \log \rho$ . Throughout this paper, we take the logarithm to base 2 and all Hilbert spaces considered are finite-dimensional.

The quantum information spectrum approach requires the extensive use of spectral projections. Any self-adjoint operator  $A$  acting on a finite-dimensional Hilbert space may be written in its spectral decomposition  $A = \sum_i \lambda_i |i\rangle\langle i|$ . We define the positive spectral projection on  $A$  as  $\{A \geq 0\} := \sum_{\lambda_i \geq 0} |i\rangle\langle i|$ , the projector onto the eigenspace of  $A$  corresponding to positive eigenvalues. Corresponding definitions apply for the other spec-

tral projections  $\{A < 0\}$ ,  $\{A > 0\}$  and  $\{A \leq 0\}$ . For two operators  $A$  and  $B$ , we can then define  $\{A \geq B\}$  as  $\{A - B \geq 0\}$ . The following key lemmas are useful. For a proof of Lemma 1, see [16, 18].

**Lemma 1** *For self-adjoint operators  $A$ ,  $B$  and any positive operator  $0 \leq P \leq I$  the inequality we have*

$$\text{Tr}[P(A - B)] \leq \text{Tr}[\{A \geq B\}(A - B)] \quad (1)$$

$$\text{Tr}[P(A - B)] \geq \text{Tr}[\{A \leq B\}(A - B)]. \quad (2)$$

*Identical conditions hold for strict inequalities in the spectral projections  $\{A < B\}$  and  $\{A > B\}$ .*

**Lemma 2** *Given a state  $\rho_n$  and a self-adjoint operator  $\omega_n$ , for any real  $\gamma$  we have*

$$\text{Tr}[\{\rho_n \geq 2^{-n\gamma} \omega_n\} \omega_n] \leq 2^{n\gamma}.$$

*Proof* Note that

$$\text{Tr}[\{\rho_n \geq 2^{-n\gamma} \omega_n\}(\rho_n - 2^{-n\gamma} \omega_n)] \geq 0$$

Hence,

$$\begin{aligned} 2^{-n\gamma} \text{Tr}[\{\rho_n \geq 2^{-n\gamma} \omega_n\} \omega_n] &\leq \text{Tr}[\{\rho_n \geq 2^{-n\gamma} \omega_n\} \rho_n] \\ &\leq \text{Tr} \rho_n = 1 \end{aligned} \quad (3)$$

Therefore,

$$\text{Tr}[\{\rho_n \geq 2^{-n\gamma} \omega_n\} \omega_n] \leq 2^{n\gamma}.$$

■

The trace distance between two operators  $A$  and  $B$  is given by

$$\|A - B\|_1 := \text{Tr}[\{A \geq B\}(A - B)] - \text{Tr}[\{A < B\}(A - B)] \quad (4)$$

The fidelity of states  $\rho$  and  $\rho'$  is defined to be

$$F(\rho, \rho') := \text{Tr} \sqrt{\rho^{\frac{1}{2}} \rho' \rho^{\frac{1}{2}}}.$$

The trace distance between two states  $\rho$  and  $\rho'$  is related to the fidelity  $F(\rho, \rho')$  as follows (see (9.110) of [17]):

$$\frac{1}{2} \|\rho - \rho'\|_1 \leq \sqrt{1 - F(\rho, \rho')^2} \leq \sqrt{2(1 - F(\rho, \rho'))}. \quad (5)$$

We also use the following simple corollary of Lemma 1:

**Corollary 1** *For self-adjoint operators  $A$ ,  $B$  and any positive operator  $0 \leq P \leq I$ , the inequality*

$$\|A - B\|_1 \leq \varepsilon,$$

*for any  $\varepsilon > 0$ , implies that*

$$\text{Tr}[P(A - B)] \leq \varepsilon.$$

We also use the “gentle measurement” lemma [19, 29].

**Lemma 3** *For a state  $\rho$  and operator  $0 \leq \Lambda \leq I$ , if  $\text{Tr}(\rho \Lambda) \geq 1 - \delta$ , then*

$$\|\rho - \sqrt{\Lambda} \rho \sqrt{\Lambda}\|_1 \leq 2\sqrt{\delta}.$$

*The same holds if  $\rho$  is only a subnormalized density operator.*

### Definition of spectral divergence rates

In the quantum information spectrum approach one defines spectral divergence rates, defined below, which can be viewed as generalizations of the quantum relative entropy.

**Definition 1** Given a sequence of states  $\hat{\rho} = \{\rho_n\}_{n=1}^\infty$  and a sequence of positive operators  $\hat{\omega} = \{\omega_n\}_{n=1}^\infty$ , the quantum spectral sup-(inf-)divergence rates are defined in terms of the difference operators  $\Pi_n(\gamma) = \rho_n - 2^{-n\gamma}\omega_n$  as

$$\overline{D}(\hat{\rho}||\hat{\omega}) := \inf \left\{ \gamma : \limsup_{n \rightarrow \infty} \text{Tr}[\{\Pi_n(\gamma) \geq 0\}\Pi_n(\gamma)] = 0 \right\} \quad (6)$$

$$\underline{D}(\hat{\rho}||\hat{\omega}) := \sup \left\{ \gamma : \liminf_{n \rightarrow \infty} \text{Tr}[\{\Pi_n(\gamma) \geq 0\}\Pi_n(\gamma)] = 1 \right\} \quad (7)$$

respectively.

Although the use of sequences of states allows for immense freedom in choosing them, there remain a number of basic properties of the quantum spectral divergence rates that hold for all sequences. These are stated and proved in [3]. In the i.i.d. case the sequence is generated from product states  $\rho = \{\varrho^{\otimes n}\}_{n=1}^\infty$ , which is used to relate the spectral entropy rates for the sequence  $\rho$  to the entropy of a single state  $\varrho$ .

Note that the above definitions of the spectral divergence rates differ slightly from those originally given in (38) and (39) of [13]. However, they are equivalent, as stated in the following two propositions (proved in [3]). The proofs have been included in the Appendix for completeness.

**Proposition 1** The spectral sup-divergence rate  $\overline{D}(\rho||\omega)$  is equal to

$$\overline{D}(\rho||\omega) = \inf \left\{ \alpha : \limsup_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\rho_n] = 0 \right\} \quad (8)$$

which is the previously used definition of the spectral sup-divergence rate. Hence the two definitions are equivalent.

**Proposition 2** The spectral inf-divergence rate  $\underline{D}(\rho||\omega)$  is equivalent to

$$\underline{D}(\rho||\omega) = \sup \left\{ \alpha : \liminf_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\rho_n] = 1 \right\} \quad (9)$$

which is the previously used definition of the spectral inf-divergence rate.

Despite these equivalences, it is useful to use the definitions (6) and (7) for the divergence rates as they allow the application of Lemmas 1 and 2 in deriving various properties of these rates.

The spectral generalizations of the von Neumann entropy, the conditional entropy and the mutual information can all be expressed as spectral divergence rates with appropriate substitutions for the sequence of operators  $\hat{\omega} = \{\omega_n\}_{n=1}^\infty$ .

### Definition of spectral entropy rates

Consider a sequence of Hilbert spaces  $\{\mathcal{H}_n\}_{n=1}^\infty$ , with  $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ . For any sequence of states  $\hat{\rho} = \{\rho_n\}_{n=1}^\infty$ , with  $\rho_n$  being a density matrix acting in the Hilbert space  $\mathcal{H}_n$ , the sup- and inf- spectral entropy rates are defined as follows:

$$\begin{aligned} \overline{S}(\hat{\rho}) &= \inf \left\{ \gamma : \liminf_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq 2^{-n\gamma}I_n\}\rho_n] = 1 \right\} \quad (10) \\ \underline{S}(\hat{\rho}) &= \sup \left\{ \gamma : \limsup_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq 2^{-n\gamma}I_n\}\rho_n] = 0 \right\}. \end{aligned} \quad (11)$$

Here  $I_n$  denotes the identity operator acting in  $\mathcal{H}_n$ . These are obtainable from the spectral divergence rates as follows [see [3]]:

$$\overline{S}(\hat{\rho}) = -\underline{D}(\hat{\rho}||\hat{I}); \quad \underline{S}(\hat{\rho}) = -\overline{D}(\hat{\rho}||\hat{I}), \quad (12)$$

where  $\hat{I} = \{I_n\}_{n=1}^\infty$  is a sequence of identity operators.

It is known [3] that the spectral entropy rates of  $\hat{\rho}$  are related to the von Neumann entropies of the states  $\rho_n$  as follows:

$$\underline{S}(\hat{\rho}) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} S(\rho_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} S(\rho_n) \leq \overline{S}(\hat{\rho}). \quad (13)$$

Moreover for a sequence of states  $\hat{\rho} = \{\rho^{\otimes n}\}_{n=1}^\infty$ :

$$\underline{S}(\hat{\rho}) = \lim_{n \rightarrow \infty} \frac{1}{n} S(\rho_n) = \overline{S}(\hat{\rho}). \quad (14)$$

For sequences of bipartite states  $\hat{\rho} = \{\rho_n^{AB}\}_{n=1}^\infty$ , with  $\rho_n^{AB} \in \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$ , the conditional spectral entropy rates are defined as follows:

$$\overline{S}(A|B) := -\underline{D}(\hat{\rho}^{AB}|\hat{I}^A \otimes \hat{\rho}^B); \quad (15)$$

$$\underline{S}(A|B) := -\overline{D}(\hat{\rho}^{AB}|\hat{I}^A \otimes \hat{\rho}^B). \quad (16)$$

In the above,  $\hat{I}^A = \{I_n^A\}_{n=1}^\infty$  and  $\hat{\rho}^A = \{\rho_n^A\}_{n=1}^\infty$ , with  $I_n^A$  being the identity operator acting in  $\mathcal{H}_A^{\otimes n}$  and  $\rho_n^A = \text{Tr}_B \rho_n^{AB}$ , the partial trace being taken on the Hilbert space  $\mathcal{H}_B^{\otimes n}$ .

### Definition of min- and max-entropies

We start with the definition of *non-smooth* min- and max-entropies.

**Definition 2 ([20])** The min- and max-entropies of a bipartite state  $\rho_{AB}$  relative to a state  $\sigma_B$  are defined by

$$H_{\min}(\rho_{AB}|\sigma_B) := -\log \min\{\lambda : \rho_{AB} \leq \lambda \cdot I_A \otimes \rho_B\}$$

and

$$H_{\max}(\rho_{AB}|\sigma_B) := \log \text{Tr}(\pi_{AB}(I_A \otimes \sigma_B)),$$

where  $\pi_{AB}$  denotes the projector onto the support of  $\rho_{AB}$ .

In the special case where the system  $B$  is trivial (i.e., 1-dimensional), we simply write  $H_{\min}(\rho_A)$  and  $H_{\max}(\rho_A)$ . These entropies then correspond to the usual *non-conditional* Rényi entropies of order infinity and zero,

$$\begin{aligned} H_{\min}(\rho_A) &= H_{\infty}(\rho_A) = -\log \|\rho_A\|_{\infty} \\ H_{\max}(\rho_A) &= H_0(\rho_A) = \log \text{rank}(\rho_A), \end{aligned}$$

where  $\|\cdot\|_{\infty}$  denotes the  $L_{\infty}$ -norm.

### Definition of smooth min- and max-entropies

*Smooth* min- and max-entropies are generalizations of the above entropy measures, involving an additional *smoothness* parameter  $\varepsilon \geq 0$ . For  $\varepsilon = 0$ , they reduce to the *non-smooth* quantities.

**Definition 3 ([20])** For any  $\varepsilon \geq 0$ , the  $\varepsilon$ -smooth min- and max-entropies of a bipartite state  $\rho_{AB}$  relative to a state  $\sigma_B$  are defined by

$$H_{\min}^{\varepsilon}(\rho_{AB}|\sigma_B) := \sup_{\bar{\rho} \in B^{\varepsilon}} H_{\min}(\bar{\rho}|\sigma_B)$$

and

$$H_{\max}^{\varepsilon}(\rho_{AB}|\sigma_B) := \inf_{\bar{\rho} \in B^{\varepsilon}} H_{\max}(\bar{\rho}|\sigma_B)$$

where  $B^{\varepsilon}(\rho) := \{\bar{\rho} \geq 0 : \|\bar{\rho} - \rho\|_1 \leq \varepsilon, \text{Tr}(\bar{\rho}) \leq \text{Tr}(\rho)\}$ .

In the following, we will focus on the smooth min- and max-entropies for the case where  $\sigma_B = \rho_B$ . Note that the quantities  $H_{\min}^{\varepsilon}(\rho_{AB}|B) := \max_{\sigma_B} H_{\min}^{\varepsilon}(\rho_{AB}|\sigma_B)$  and  $H_{\max}^{\varepsilon}(\rho_{AB}|B) := \min_{\sigma_B} H_{\max}^{\varepsilon}(\rho_{AB}|\sigma_B)$  defined in [20] are not studied in this paper.

### RELATION BETWEEN NON-CONDITIONAL ENTROPIES

#### Relation between $\underline{S}(\hat{\rho})$ and $H_{\min}^{\varepsilon}(\rho)$

**Theorem 1** Given a sequence of states  $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$ , where  $\rho_n \in \mathcal{B}(\mathcal{H}_n)$ , with  $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ , the inf-spectral entropy rate  $\underline{S}(\hat{\rho})$  is related to the smooth min-entropy as follows:

$$\underline{S}(\hat{\rho}) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^{\varepsilon}(\rho_n) \quad (17)$$

*Proof* For any constant  $\gamma > 0$ , let us define projection operators

$$Q_n^{\gamma} := \{\rho_n < 2^{-n\gamma} I_n\} \quad (18)$$

and

$$P_n^{\gamma} := I_n - Q_n^{\gamma} = \{\rho_n \geq 2^{-n\gamma} I_n\}. \quad (19)$$

In terms of these projections, we can write

$$\underline{S}(\hat{\rho}) = \sup \left\{ \gamma : \limsup_{n \rightarrow \infty} \text{Tr}[P_n^{\gamma} \rho_n] = 0 \right\}, \quad (20)$$

or alternatively as

$$\underline{S}(\hat{\rho}) = \sup \left\{ \gamma : \liminf_{n \rightarrow \infty} \text{Tr}[Q_n^{\gamma} \rho_n] = 1 \right\}, \quad (21)$$

since each  $\rho_n$  in the sequence  $\hat{\rho}$  is a state (i.e.,  $\text{Tr} \rho_n = 1$ ). From Proposition 2 and (12) of  $\underline{S}(\hat{\rho})$  it follows that the latter is equivalently given by the expression

$$\underline{S}(\hat{\rho}) = \sup \left\{ \gamma : \limsup_{n \rightarrow \infty} \text{Tr}[P_n^{\gamma}(\rho_n - 2^{-n\gamma} I_n)] = 0 \right\}, \quad (22)$$

From (21) it follows that, for any  $\gamma < \underline{S}(\hat{\rho})$  and any  $\delta > 0$ , for  $n$  large enough,

$$\text{Tr}[Q_n^{\gamma} \rho_n] > 1 - \delta. \quad (23)$$

For any given  $\alpha > 0$ , let  $\gamma := \underline{S}(\hat{\rho}) - \alpha$ , and let

$$\tilde{\rho}_n^{\gamma} := Q_n^{\gamma} \rho_n Q_n^{\gamma} \quad (24)$$

Then using (23) and Lemma 3 we infer that, for  $n$  large enough,

$$\|\rho_n - \tilde{\rho}_n^{\gamma}\|_1 \leq 2\sqrt{\delta}. \quad (25)$$

In other words, for  $n$  large enough,  $\tilde{\rho}_n^{\gamma} \in B^{\varepsilon}(\rho_n)$  with  $\varepsilon = 2\sqrt{\delta}$ .

We first prove the upper bound

$$\underline{S}(\hat{\rho}) \leq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^{\varepsilon}(\rho_n) \quad (26)$$

For  $n$  large enough,

$$\begin{aligned} H_{\min}^{\varepsilon}(\rho_n) &\equiv \sup_{\bar{\rho}_n \in B^{\varepsilon}(\rho_n)} H_{\min}(\bar{\rho}_n) \\ &\geq H_{\min}(\tilde{\rho}_n^{\gamma}) = -\log \|\tilde{\rho}_n^{\gamma}\|_{\infty} \\ &> n\gamma = n(\underline{S}(\hat{\rho}) - \alpha) \end{aligned} \quad (27)$$

The last line follows from the inequality  $\tilde{\rho}_n^{\gamma} < 2^{-n\gamma} I_n$ , and since  $\alpha$  is arbitrary, we obtain the desired bound (26).

We next prove the converse, i.e.,

$$\underline{S}(\hat{\rho}) \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^{\varepsilon}(\rho_n) \quad (28)$$

Consider an operator  $\bar{\rho}_n^\varepsilon \in B^\varepsilon(\rho_n)$  for which

$$-\log \|\bar{\rho}_n^\varepsilon\|_\infty = \sup_{\bar{\rho}_n \in B^\varepsilon(\rho_n)} [-\log \|\bar{\rho}_n\|_\infty]. \quad (29)$$

We shall also make use of a quantity  $\underline{\Upsilon}(\hat{\omega})$ , defined for any sequence of positive operators  $\hat{\omega} = \{\omega_n\}_{n=1}^\infty$  as follows:

$$\underline{\Upsilon}(\hat{\omega}) = \sup \left\{ \alpha : \limsup_{n \rightarrow \infty} \text{Tr}[\{\omega_n \geq 2^{-n\alpha} I_n\} \Pi_n^\alpha] = 0 \right\}, \quad (30)$$

where  $\Pi_n^\alpha := (\omega_n - 2^{-n\alpha} I_n)$ . Note that  $\underline{\Upsilon}(\hat{\omega})$  reduces to the inf-spectral entropy rate  $\underline{\mathcal{S}}(\hat{\omega})$  given by (22), if  $\hat{\omega}$  is a sequence of states.

By the definition of the smooth min-entropy, (28) then follows from Lemma 4 below.  $\blacksquare$

**Lemma 4** *For any sequence of states  $\hat{\rho} = \{\rho_n\}_{n=1}^\infty$ , and any  $\varepsilon > 0$ , there exists an  $n_0 \in \mathbb{N}$ , such that for all  $n \geq n_0$*

$$\underline{\mathcal{S}}(\hat{\rho}) \geq \frac{1}{n} [-\log \|\bar{\rho}_n^\varepsilon\|_\infty], \quad (31)$$

with  $\bar{\rho}_n^\varepsilon$  defined by (29).

*Proof* We prove this in two steps. We first prove that for any  $\varepsilon > 0$  and  $n$  large enough,

$$\underline{\Upsilon}(\hat{\rho}^\varepsilon) \geq -\frac{1}{n} \log \|\bar{\rho}_n^\varepsilon\|_\infty, \quad (32)$$

where  $\hat{\rho}^\varepsilon := \{\bar{\rho}_n^\varepsilon\}_{n=1}^\infty$ . We then prove that

$$\lim_{\varepsilon \rightarrow 0} \underline{\Upsilon}(\hat{\rho}^\varepsilon) \leq \underline{\mathcal{S}}(\hat{\rho}) \quad (33)$$

For any arbitrary  $\eta > 0$ , let  $\alpha$  be defined through the relation

$$\|\bar{\rho}_n^\varepsilon\|_\infty = 2^{-n(\alpha+\eta)}. \quad (34)$$

This implies the operator inequality,  $\bar{\rho}_n^\varepsilon - 2^{-n(\alpha+\eta)} I_n \leq 0$ , and hence  $\bar{\rho}_n^\varepsilon < 2^{-n\alpha} I_n$ .

Hence,

$$\text{Tr}[\{\bar{\rho}_n^\varepsilon \geq 2^{-n\alpha} I_n\} (\bar{\rho}_n^\varepsilon - 2^{-n\alpha} I_n)] = 0, \quad (35)$$

Using this, and the definition of  $\underline{\Upsilon}(\hat{\rho}^\varepsilon)$ , we infer that  $\alpha \leq \underline{\Upsilon}(\hat{\rho}^\varepsilon)$ . Then, using (34) we obtain the bound

$$-\frac{1}{n} \log \|\bar{\rho}_n^\varepsilon\|_\infty - \eta \leq \underline{\Upsilon}(\hat{\rho}^\varepsilon),$$

which in turn yields (32), since  $\eta$  is arbitrary.

To prove (33) note that

$$\begin{aligned} 0 &\leq \text{Tr}(P_n^\gamma \rho_n) \\ &= \text{Tr}(P_n^\gamma \bar{\rho}_n^\varepsilon) + \text{Tr}[P_n^\gamma (\rho_n - \bar{\rho}_n^\varepsilon)] \\ &\leq \text{Tr}[P_n^\gamma (\bar{\rho}_n^\varepsilon - 2^{-n\alpha} I_n)] + 2^{-n\alpha} \text{Tr} P_n^\gamma + \varepsilon \\ &\leq \text{Tr}[\{\bar{\rho}_n^\varepsilon \geq 2^{-n\alpha} I_n\} (\bar{\rho}_n^\varepsilon - 2^{-n\alpha} I_n)] + 2^{-n(\alpha-\gamma)} + \varepsilon. \end{aligned} \quad (36)$$

The third line in (36) is obtained by using the bound

$$\text{Tr}[P_n^\gamma (\rho_n - \bar{\rho}_n^\varepsilon)] \leq \varepsilon,$$

which follows from Corollary 1, since  $\bar{\rho}_n^\varepsilon \in B^\varepsilon(\rho_n)$ .

To arrive at the last line of (36) we use Lemma 1 and the fact that  $\text{Tr} P_n^\gamma \leq 2^{n\gamma}$ , which follows from Lemma 2.

Let us choose  $\gamma = \alpha - \delta/2$ , for an arbitrary  $\delta > 0$ , with  $\alpha = \underline{\Upsilon}(\hat{\rho}^\varepsilon) - \delta/2$ . Then both the first and second terms on the r.h.s. of (36) goes to zero as  $n \rightarrow \infty$ . Therefore, for  $n$  large enough and any  $\delta' > 0$ , in the limit  $\varepsilon \rightarrow 0$ , we must have that

$$\text{Tr}(P_n^\gamma \rho_n) \leq \delta', \quad (37)$$

which in turn implies that  $\gamma \leq \underline{\mathcal{S}}(\hat{\rho})$ .

From the choice of the parameters  $\alpha$  and  $\gamma$  it follows that

$$\lim_{\varepsilon \rightarrow 0} \underline{\Upsilon}(\hat{\rho}^\varepsilon) - \delta < \underline{\mathcal{S}}(\hat{\rho}). \quad (38)$$

But since  $\delta$  is arbitrary, we obtain the inequality (33).  $\blacksquare$

#### Relation between $\overline{\mathcal{S}}(\hat{\rho})$ and $H_{\max}^\varepsilon(\rho)$

**Theorem 2** *Given a sequence of states  $\hat{\rho} = \{\rho_n\}_{n=1}^\infty$ , where  $\rho_n \in \mathcal{B}(\mathcal{H}_n)$ , with  $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ , the sup-spectral entropy rate  $\overline{\mathcal{S}}(\hat{\rho})$  is related to the smooth max-entropy as follows:*

$$\overline{\mathcal{S}}(\hat{\rho}) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(\rho_n) \quad (39)$$

*Proof* By definition, the sup-spectral entropy rate for the given sequence of states is

$$\overline{\mathcal{S}}(\hat{\rho}) = \inf \left\{ \gamma : \liminf_{n \rightarrow \infty} \text{Tr}[P_n^\gamma \rho_n] = 1 \right\}, \quad (40)$$

where  $P_n^\gamma$  is the projection operator defined by (19).

From (40) it follows that, for any  $\gamma \geq \overline{\mathcal{S}}(\hat{\rho})$  and any  $\delta > 0$ , for  $n$  large enough

$$\text{Tr}[P_n^\gamma \rho_n] > 1 - \delta. \quad (41)$$

For any given  $\alpha > 0$ , choose  $\gamma = \overline{\mathcal{S}}(\hat{\rho}) + \alpha$ , and let

$$\tilde{\rho}_n^\gamma := P_n^\gamma \rho_n P_n^\gamma \quad (42)$$

Then using (41) and Lemma 3 we infer that, for  $n$  large enough,

$$\|\rho_n - \tilde{\rho}_n^\gamma\|_1 \leq 2\sqrt{\delta}. \quad (43)$$

and hence  $\tilde{\rho}_n^\gamma \in B^\varepsilon(\rho_n)$  with  $\varepsilon = 2\sqrt{\delta}$ .

We first prove the bound

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(\rho_n) \leq \overline{\mathcal{S}}(\hat{\rho}) \quad (44)$$

For  $n$  large enough,

$$\begin{aligned} H_{\max}^\varepsilon(\rho_n) &= \inf_{\bar{\rho}_n \in B^\varepsilon(\rho_n)} H_{\max}(\bar{\rho}_n) \\ &\leq H_{\max}(\tilde{\rho}_n^\gamma) \\ &= \log \text{rank}(\tilde{\rho}_n^\gamma) \end{aligned} \quad (45)$$

From the definition (42) of  $\tilde{\rho}_n^\gamma$  it follows that  $\text{rank} \tilde{\rho}_n^\gamma \leq \text{Tr} P_n^\gamma$ . Hence,

$$\begin{aligned} H_{\max}^\varepsilon(\rho_n) &\leq \log \text{Tr} P_n^\gamma \\ &\leq n\gamma = \bar{S}(\hat{\rho}) + \alpha, \end{aligned} \quad (46)$$

where once again we use the bound  $\text{Tr} P_n^\gamma \leq 2^{n\gamma}$ . The last line of (46) yields the desired bound (44) since  $\alpha$  is arbitrary.

To complete the proof of Theorem 2 we assume that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(\rho_n) < \bar{S}(\hat{\rho}) \quad (47)$$

and show that this leads to a contradiction.

Let  $\sigma_{n,\varepsilon}$  be the operator for which

$$H_{\max}(\sigma_{n,\varepsilon}) := \inf_{\bar{\rho}_n \in B^\varepsilon(\rho_n)} H_{\max}(\bar{\rho}_n). \quad (48)$$

Hence,  $H_{\max}^\varepsilon(\rho_n) = \log \text{rank} \sigma_{n,\varepsilon}$ , and the assumption (47) is equivalent to the following assumption:

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{rank} \sigma_{n,\varepsilon} < \bar{S}(\hat{\rho}). \quad (49)$$

Since  $\sigma_{n,\varepsilon} \in B^\varepsilon(\rho_n)$ ,  $\text{Tr} \sigma_{n,\varepsilon} \geq 1 - \varepsilon$ . Let  $\sigma_{n,\varepsilon}^0$  denote the projection onto the support of  $\sigma_{n,\varepsilon}$ . Then

$$\begin{aligned} \text{Tr}(\sigma_{n,\varepsilon}^0 \rho_n) &= \text{Tr}[(\rho_n - \sigma_{n,\varepsilon}) + \sigma_{n,\varepsilon} \sigma_{n,\varepsilon}^0] \\ &= \text{Tr}[(\rho_n - \sigma_{n,\varepsilon}) \sigma_{n,\varepsilon}^0] + \text{Tr} \sigma_{n,\varepsilon} \\ &\geq \text{Tr}[\{\rho_n \leq \sigma_{n,\varepsilon}\}(\rho_n - \sigma_{n,\varepsilon})] + 1 - \varepsilon \\ &\geq -\varepsilon + 1 - \varepsilon = 1 - 2\varepsilon. \end{aligned} \quad (50)$$

The inequality in the third line follows from Lemma 1. We arrive at the last inequality in (50) by using the bound

$$\text{Tr}[\{\rho_n \leq \sigma_{n,\varepsilon}\}(\rho_n - \sigma_{n,\varepsilon})] \geq -\varepsilon,$$

which arises from the fact that  $\sigma_{n,\varepsilon} \in B^\varepsilon(\rho_n)$ .

Note, however, that for  $n$  large enough, (50) leads to a contradiction, in the limit  $\varepsilon \rightarrow 0$ . This is because, for any real number  $R < \bar{S}(\hat{\rho})$  and any projection  $\pi_n$ , with  $\text{Tr} \pi_n = 2^{nR}$ , for  $n$  large enough, we have

$$\text{Tr}(\pi_n \rho_n) \leq 1 - c_0, \quad (51)$$

for some constant  $c_0 > 0$ . The inequality (51) can be proved as follows:

$$\begin{aligned} \text{Tr}(\pi_n \rho_n) &= \text{Tr}[\pi_n(\rho_n - 2^{-n\beta} I_n)] + 2^{-n\beta} \text{Tr} \pi_n \\ &\leq \text{Tr}[\{\rho_n \geq 2^{-n\beta} I_n\}(\rho_n - 2^{-n\beta} I_n)] \\ &\quad + 2^{-n(\beta-R)} \end{aligned} \quad (52)$$

Choose  $\bar{S}(\hat{\rho}) > \beta > R$ . For such a choice, the second term on the right hand side of (52) tends to zero asymptotically in  $n$ . However, the first term does not tend to 1 and we hence obtain the bound (51).  $\blacksquare$

## RELATION BETWEEN CONDITIONAL ENTROPIES

Consider a sequence of bipartite states  $\hat{\rho}^{AB} = \{\rho_n^{AB}\}_{n=1}^\infty$ , with  $\rho_n^{AB} \in \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$ . Let  $\hat{\rho}^{AB} = \{\rho_n^{AB}\}_{n=1}^\infty$  denote the corresponding sequence of reduced states.

For the sequence  $\hat{\rho}^{AB}$ , the sup-spectral conditional entropy rate  $\bar{S}(A|B)$  and the inf-spectral conditional entropy rate  $\underline{S}(A|B)$ , defined respectively by (15) and (16), can be expressed as follows:

$$\bar{S}(A|B) = \inf \left\{ \gamma : \liminf_{n \rightarrow \infty} \text{Tr}[P_n^\gamma \rho_n^{AB}] = 1 \right\}, \quad (53)$$

$$\underline{S}(A|B) = \sup \left\{ \gamma : \limsup_{n \rightarrow \infty} \text{Tr}[P_n^\gamma \rho_n^{AB}] = 0 \right\}, \quad (54)$$

where

$$P_n^\gamma := \{\rho_n^{AB} \geq 2^{-n\gamma} I_n^A \otimes \rho_n^B\}. \quad (55)$$

Here  $I_n^A$  denotes the identity operator in  $\mathcal{B}(\mathcal{H}_A^{\otimes n})$ .

We use the following key properties of  $H_{\min}^\varepsilon(\rho_{AB}|\rho_B)$  given by Lemma 5 and Lemma 6 below.

**Lemma 5** *Let  $\rho_{AB}$  and  $\sigma_B$  be density operators, let  $\Delta_{AB}$  be a positive operator, and let  $\lambda \in \mathbb{R}$  such that*

$$\rho_{AB} \leq 2^{-\lambda} \cdot I_A \otimes \sigma_B + \Delta_{AB}.$$

*Then  $H_{\min}^\varepsilon(\rho_{AB}|\sigma_B) \geq \lambda$  for any  $\varepsilon \geq \sqrt{8\text{Tr}(\Delta_{AB})}$ .*

*Proof* Define

$$\begin{aligned} \alpha_{AB} &:= 2^{-\lambda} \cdot I_A \otimes \sigma_B \\ \beta_{AB} &:= 2^{-\lambda} \cdot I_A \otimes \sigma_B + \Delta_{AB}. \end{aligned}$$

and

$$T_{AB} := \alpha_{AB}^{\frac{1}{2}} \beta_{AB}^{-\frac{1}{2}}.$$

Let  $|\Psi\rangle = |\Psi\rangle_{ABR}$  be a purification of  $\rho_{AB}$  and let  $|\Psi'\rangle := T_{AB} \otimes I_R |\Psi\rangle$  and  $\rho'_{AB} := \text{Tr}_R(|\Psi'\rangle\langle\Psi'|)$ .

Note that

$$\begin{aligned} \rho'_{AB} &= T_{AB} \rho_{AB} T_{AB}^\dagger \\ &\leq T_{AB} \beta_{AB} T_{AB}^\dagger \\ &= \alpha_{AB} = 2^{-\lambda} \cdot I_A \otimes \sigma_B, \end{aligned}$$

which implies  $H_{\min}(\rho'_{AB}|\sigma_B) \geq \lambda$ . It thus remains to be shown that

$$\|\rho_{AB} - \rho'_{AB}\|_1 \leq \sqrt{8\text{Tr}(\Delta_{AB})}. \quad (56)$$

We first show that the Hermitian operator

$$\bar{T}_{AB} := \frac{1}{2}(T_{AB} + T_{AB}^\dagger) .$$

satisfies

$$\bar{T}_{AB} \leq I_{AB} . \quad (57)$$

For any vector  $|\phi\rangle = |\phi\rangle_{AB}$ ,

$$\begin{aligned} \|T_{AB}|\phi\rangle\|^2 &= \langle\phi|T_{AB}^\dagger T_{AB}|\phi\rangle = \langle\phi|\beta_{AB}^{-\frac{1}{2}}\alpha_{AB}\beta_{AB}^{-\frac{1}{2}}|\phi\rangle \\ &\leq \langle\phi|\beta_{AB}^{-\frac{1}{2}}\beta_{AB}\beta_{AB}^{-\frac{1}{2}}|\phi\rangle = \|\phi\|^2 \end{aligned}$$

where the inequality follows from  $\alpha_{AB} \leq \beta_{AB}$ . Similarly,

$$\begin{aligned} \|T_{AB}^\dagger|\phi\rangle\|^2 &= \langle\phi|T_{AB}T_{AB}^\dagger|\phi\rangle = \langle\phi|\alpha_{AB}^{\frac{1}{2}}\beta_{AB}^{-1}\alpha_{AB}^{\frac{1}{2}}|\phi\rangle \\ &\leq \langle\phi|\alpha_{AB}^{\frac{1}{2}}\alpha_{AB}^{-1}\alpha_{AB}^{\frac{1}{2}}|\phi\rangle = \|\phi\|^2 \end{aligned}$$

where the inequality follows from the fact that  $\beta_{AB}^{-1} \leq \alpha_{AB}^{-1}$  which holds because the function  $\tau \mapsto -\tau^{-1}$  is operator monotone on  $(0, \infty)$  (see Proposition V.1.6 of [2]). We conclude that for any vector  $|\phi\rangle$ ,

$$\begin{aligned} \|\bar{T}_{AB}|\phi\rangle\| &\leq \frac{1}{2}\|T_{AB}|\phi\rangle + T_{AB}^\dagger|\phi\rangle\| \\ &\leq \frac{1}{2}\|T_{AB}|\phi\rangle\| + \frac{1}{2}\|T_{AB}^\dagger|\phi\rangle\| \leq \|\phi\| , \end{aligned}$$

which implies (57).

We now determine the overlap between  $|\Psi\rangle$  and  $|\Psi'\rangle$ ,

$$\begin{aligned} \langle\Psi|\Psi'\rangle &= \langle\Psi|T_{AB} \otimes I_R|\Psi'\rangle \\ &= \text{Tr}(|\Psi\rangle\langle\Psi|T_{AB} \otimes I_R) = \text{Tr}(\rho_{AB}T_{AB}) . \end{aligned}$$

Because  $\rho_{AB}$  has trace one, we have

$$\begin{aligned} 1 - |\langle\Psi|\Psi'\rangle| &\leq 1 - \Re\langle\Psi|\Psi'\rangle = \text{Tr}(\rho_{AB}(I_{AB} - \bar{T}_{AB})) \\ &\leq \text{Tr}(\beta_{AB}(I_{AB} - \bar{T}_{AB})) \\ &= \text{Tr}(\beta_{AB}) - \text{Tr}(\alpha_{AB}^{\frac{1}{2}}\beta_{AB}^{\frac{1}{2}}) \\ &\leq \text{Tr}(\beta_{AB}) - \text{Tr}(\alpha_{AB}) = \text{Tr}(\Delta_{AB}) . \end{aligned}$$

Here, the second inequality follows from the fact that, because of (57), the operator  $I_{AB} - \bar{T}_{AB}$  is positive and  $\rho_{AB} \leq \beta_{AB}$ . The last inequality holds because  $\alpha_{AB}^{\frac{1}{2}} \leq \beta_{AB}^{\frac{1}{2}}$ , which is a consequence of the operator monotonicity of the square root (Proposition V.1.8 of [2]).

Using (5) and the fact that the fidelity between two pure states is given by their overlap, we find

$$\begin{aligned} \|\Psi\rangle\langle\Psi| - |\Psi'\rangle\langle\Psi'|\|_1 &\leq 2\sqrt{2(1 - |\langle\Psi|\Psi'\rangle|)} \\ &\leq 2\sqrt{2\text{Tr}(\Delta_{AB})} \leq \varepsilon . \end{aligned}$$

Inequality (56) then follows because the trace distance can only decrease when taking the partial trace.  $\blacksquare$

**Lemma 6** Let  $\rho_{AB}$  and  $\sigma_B$  be density operators. Then

$$H_{\min}^\varepsilon(\rho_{AB}|\sigma_B) \geq \lambda$$

for any  $\lambda \in \mathbb{R}$  and

$$\varepsilon = \sqrt{8\text{Tr}(\{\rho_{AB} > 2^{-\lambda} \cdot I_A \otimes \sigma_B\}\rho_{AB})} .$$

*Proof* Let  $\Delta_{AB}^+$  and  $\Delta_{AB}^-$  be mutually orthogonal positive operators such that

$$\Delta_{AB}^+ - \Delta_{AB}^- = \rho_{AB} - 2^{-\lambda} \cdot I_A \otimes \sigma_B .$$

Furthermore, let  $P_{AB}$  be the projector onto the support of  $\Delta_{AB}^+$ , i.e.,

$$P_{AB} = \{\rho_{AB} > 2^{-\lambda} \cdot I_A \otimes \sigma_B\} .$$

We then have

$$\begin{aligned} P_{AB}\rho_{AB}P_{AB} &= P_{AB}(2^{-\lambda} \cdot I_A \otimes \sigma_B + \Delta_{AB}^+ - \Delta_{AB}^-)P_{AB} \\ &\geq \Delta_{AB}^+ \end{aligned}$$

and, hence,

$$\sqrt{8\text{Tr}(\Delta_{AB}^+)} \leq \sqrt{8\text{Tr}(P_{AB}\rho_{AB})} = \varepsilon .$$

The assertion now follows from Lemma 5 because

$$\rho_{AB} \leq 2^{-\lambda} \cdot I_A \otimes \sigma_B + \Delta_{AB}^+ .$$

$\blacksquare$

In the following sections we state and prove the relations between the conditional spectral entropy rates and the smooth conditional max- and min-entropy.

### Relation between $\bar{S}(A|B)$ and $H_{\max}^\varepsilon(\rho_{AB}|\rho_B)$

**Theorem 3** Given a sequence of bipartite states  $\hat{\rho}^{AB} = \{\rho_n^{AB}\}_{n=1}^\infty$ , where  $\rho_n^{AB} \in \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$ , the sup-spectral conditional entropy rate  $\bar{S}(A|B)$ , defined by (53), satisfies

$$\bar{S}(A|B) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(\rho_n^{AB}|\rho_n^B), \quad (58)$$

where  $H_{\max}^\varepsilon(\rho_n^{AB}|\rho_n^B)$  is the smooth max-entropy of the state  $\rho_n^{AB}$  of the sequence, conditional on the corresponding reduced state  $\rho_n^B$ .

*Proof* From the definition (53) of  $\bar{S}(A|B)$  it follows that for any  $\gamma \geq \bar{S}(A|B)$  and any  $\delta > 0$ , for  $n$  large enough

$$\text{Tr}[P_n^\gamma \rho_n^{AB}] > 1 - \delta, \quad (59)$$

where  $P_n^\gamma$  is defined by (55).

For any given  $\alpha > 0$ , choose  $\gamma = \overline{S}(A|B) + \alpha$ , and let

$$\rho_{n,\gamma}^{AB} := P_n^\gamma \rho_n^{AB} P_n^\gamma \quad (60)$$

Then using (59) and Lemma 3 we infer that, for  $n$  large enough,  $\rho_{n,\gamma}^{AB} \in B^\varepsilon(\rho_n^{AB})$  with  $\varepsilon = 2\sqrt{\delta}$ . Let  $\pi_{n,\gamma}^{AB}$  denote the projection onto the support of  $\rho_{n,\gamma}^{AB}$ .

We first prove bound

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(\rho_n) \leq \overline{S}(A|B). \quad (61)$$

For  $n$  large enough,

$$\begin{aligned} H_{\max}^\varepsilon(\rho_n^{AB} | \rho_n^B) &:= \inf_{\bar{\rho}_n \in B^\varepsilon(\rho_n^{AB})} H_{\max}(\bar{\rho}_n^{AB} | \rho_n^B) \\ &\leq H_{\max}(\rho_{n,\gamma}^{AB} | \rho_n^B) \\ &= \log \text{Tr}((I_n^A \otimes \rho_n^B) \pi_{n,\gamma}^{AB}) \\ &\leq \log \text{Tr}((I_n^A \otimes \rho_n^B) P_n^\gamma) \\ &\leq n\gamma \end{aligned} \quad (62)$$

The last inequality in (62) follows from Lemma 2. Hence, for  $n$  large enough,

$$\frac{1}{n} H_{\max}^\varepsilon(\rho_n^{AB} | \rho_n^B) \leq \gamma = \overline{S}(A|B) + \alpha, \quad (63)$$

and since  $\alpha$  is arbitrary, we obtain the desired bound (61).

To complete the proof of Theorem 3, we assume that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(\rho_n^{AB} | \rho_n^B) < \overline{S}(A|B), \quad (64)$$

and prove that this leads to a contradiction. Let  $\sigma_{n,\varepsilon}^{AB}$  be the operator for which

$$H_{\max}(\sigma_{n,\varepsilon}^{AB} | \rho_n^B) = \inf_{\bar{\rho}^{AB} \in B^\varepsilon(\rho_n^{AB})} H_{\max}(\bar{\rho}^{AB} | \rho_n^B). \quad (65)$$

Hence,

$$\begin{aligned} H_{\max}^\varepsilon(\rho_n^{AB} | \rho_n^B) &= H_{\max}(\sigma_{n,\varepsilon}^{AB} | \rho_n^B) \\ &= \log \text{Tr}((I_n^A \otimes \rho_n^B) \pi_{n,\varepsilon}^{AB}), \end{aligned} \quad (66)$$

where  $\pi_{n,\varepsilon}^{AB}$  is the projection onto the support of  $\sigma_{n,\varepsilon}^{AB}$ .

Hence, the assumption (64) is equivalent to the following assumption:

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Tr}[\pi_{n,\varepsilon}^{AB} (I_n^A \otimes \rho_n^B)] < \overline{S}(A|B). \quad (67)$$

Note that

$$\begin{aligned} &\text{Tr}(\pi_{n,\varepsilon}^{AB} \rho_n^{AB}) \\ &= \text{Tr}[(\rho_n^{AB} - \sigma_{n,\varepsilon}^{AB}) + \sigma_{n,\varepsilon}^{AB}] \pi_{n,\varepsilon}^{AB} \\ &= \text{Tr}[(\rho_n^{AB} - \sigma_{n,\varepsilon}^{AB}) \pi_{n,\varepsilon}^{AB}] + \text{Tr} \sigma_{n,\varepsilon}^{AB} \\ &\geq \text{Tr}[\{\rho_n^{AB} \leq \sigma_{n,\varepsilon}^{AB}\} (\rho_n^{AB} - \sigma_{n,\varepsilon}^{AB})] + \text{Tr}[\sigma_{n,\varepsilon}^{AB}] \\ &\geq -\varepsilon + 1 - \varepsilon = 1 - 2\varepsilon. \end{aligned} \quad (68)$$

We arrive at the second last line of (68) using Lemma 1. The last line of (68) is obtained analogously to (50), since  $\sigma_{n,\varepsilon}^{AB} \in B^\varepsilon(\rho_n^{AB})$ .

Note, however, that (68) leads to a contradiction. This can be seen as follows: Let  $R$  be a real number satisfying

$$\text{Tr}[\pi_{n,\varepsilon}^{AB} (I_n^A \otimes \rho_n^B)] = 2^{nR}.$$

It follows from the assumption (67) that, for  $\varepsilon$  small enough,  $R < \overline{S}(A|B)$ . Note that

$$\begin{aligned} &\text{Tr}(\pi_{n,\varepsilon}^{AB} \rho_n^{AB}) \\ &= \text{Tr}[\pi_{n,\varepsilon}^{AB} (\rho_n^{AB} - 2^{-n\gamma} I_n^A \otimes \rho_n^B)] \\ &\quad + 2^{-n\gamma} \text{Tr}[\pi_{n,\varepsilon}^{AB} (I_n^A \otimes \rho_n^B)] \\ &\leq \text{Tr}[\{\rho_n^{AB} \geq 2^{-n\gamma} I_n^A \otimes \rho_n^B\} (\rho_n^{AB} - 2^{-n\gamma} I_n^A \otimes \rho_n^B)] \\ &\quad + 2^{-n(\gamma-R)} \end{aligned} \quad (69)$$

Choose  $\overline{S}(A|B) > \gamma > R$ . For such a choice, the second term on the right hand side of (69) tends to zero asymptotically in  $n$ . However, the first term does not tend to 1 and we hence obtain the bound

$$\text{Tr}(\pi_{n,\varepsilon}^{AB} \rho_n^{AB}) < 1 - c_0, \quad (70)$$

for some constant  $c_0 > 0$ . This contradicts (68) in the limit  $\varepsilon \rightarrow 0$ .  $\blacksquare$

#### Relation between $\underline{S}(A|B)$ and $H_{\min}^\varepsilon(\rho_{AB} | \rho_B)$

**Theorem 4** Given a sequence of bipartite states  $\hat{\rho}^{AB} = \{\rho_n^{AB}\}_{n=1}^\infty$ , where  $\rho_n^{AB} \in \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$ , the inf-spectral conditional entropy rate  $\underline{S}(A|B)$  is related to the smooth conditional min-entropy as follows:

$$\underline{S}(A|B) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\varepsilon(\rho_n^{AB} | \rho_n^B) \quad (71)$$

*Proof*

We first prove the bound

$$\underline{S}(A|B) \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\varepsilon(\rho_n^{AB} | \rho_n^B) \quad (72)$$

Let  $\sigma_{n,\varepsilon}^{AB}$  be the operator for which

$$H_{\min}(\sigma_{n,\varepsilon}^{AB} | \rho_n^B) = \max_{\bar{\rho}^{AB} \in B^\varepsilon(\rho_n^{AB})} H_{\min}(\bar{\rho}^{AB} | \rho_n^B). \quad (73)$$

Let us define

$$\begin{aligned} &\underline{\mathcal{I}}^\varepsilon(A|B) \\ &:= \sup \left\{ \alpha : \limsup_{n \rightarrow \infty} \text{Tr}[\{\sigma_{n,\varepsilon}^{AB} \geq 2^{-n\alpha} I_n^A \otimes \rho_n^B\} \Pi_n^\alpha] = 0 \right\}, \end{aligned} \quad (74)$$

where  $\Pi_n^\alpha := \sigma_{n,\varepsilon}^{AB} - 2^{-n\alpha} I_n^A \otimes \rho_n^B$ .

According to Definition 3 of the conditional smooth min-entropy, that to prove (72), it suffices to prove the following lemma:



**Lemma 7** For any sequence of bipartite states  $\hat{\rho}^{AB} = \{\rho_n^{AB}\}_{n=1}^\infty$ , and any  $\varepsilon > 0$ , there exists an  $n_0 \in \mathbb{N}$ , such that for all  $n \geq n_0$

$$\underline{S}(A|B) \geq -\frac{1}{n} \log[\{\min\{\lambda : \sigma_{n,\varepsilon}^{AB} \leq \lambda I_n^A \otimes \rho_n^B\}\}], \quad (75)$$

with  $\sigma_{n,\varepsilon}^{AB}$  defined by (73).

*Proof* We prove this lemma in two steps. We first prove that for any  $\varepsilon > 0$  and  $n$  large enough,

$$\underline{\Upsilon}^\varepsilon(A|B) \geq -\frac{1}{n} \log[\{\min\{\lambda : \sigma_{n,\varepsilon}^{AB} \leq \lambda I_n^A \otimes \rho_n^B\}\}]. \quad (76)$$

We then prove that

$$\underline{S}(A|B) \geq \lim_{\varepsilon \rightarrow 0} \underline{\Upsilon}^\varepsilon(A|B). \quad (77)$$

*Proof of (76):* For any arbitrary  $\eta > 0$ , let  $\alpha$  be defined through the relation

$$\min\{\lambda : \sigma_{n,\varepsilon}^{AB} \leq \lambda I_n^A \otimes \rho_n^B\} = 2^{-n(\alpha+\eta)}. \quad (78)$$

Hence,

$$-\frac{1}{n} \log[\{\min\{\lambda : \sigma_{n,\varepsilon}^{AB} \leq \lambda I_n^A \otimes \rho_n^B\}\}] = \alpha + \eta \quad (79)$$

Note that (78) implies that  $\sigma_{n,\varepsilon}^{AB} \leq 2^{-n(\alpha+\eta)}(I_n^A \otimes \rho_n^B)$ , and hence  $(\sigma_{n,\varepsilon}^{AB} - 2^{-n(\alpha+\eta)}I_n^A \otimes \rho_n^B) \leq 0$ . This in turn implies that  $(\sigma_{n,\varepsilon}^{AB} - 2^{-n\alpha}I_n^A \otimes \rho_n^B) \leq 0$  and hence

$$\text{Tr}[\{\sigma_{n,\varepsilon}^{AB} \geq 2^{-n\alpha}I_n^A \otimes \rho_n^B\}(\sigma_{n,\varepsilon}^{AB} - 2^{-n\alpha}I_n^A \otimes \rho_n^B)] = 0. \quad (80)$$

It then follows from the definition (74) of  $\underline{\Upsilon}^\varepsilon(A|B)$  that  $\alpha \leq \underline{\Upsilon}^\varepsilon(A|B)$ . Hence, using (79), we get

$$-\frac{1}{n} \log[\{\min\{\lambda : \sigma_{n,\varepsilon}^{AB} \leq \lambda I_n^A \otimes \rho_n^B\}\}] - \eta \leq \underline{\Upsilon}^\varepsilon(A|B), \quad (81)$$

which in turn yields (76), since  $\eta$  is arbitrary.

*Proof of (77):* Defining  $P_n^\gamma := \{\rho_n^{AB} \geq 2^{-n\gamma}I_n^A \otimes \rho_n^B\}$ , note that

$$\begin{aligned} & \text{Tr}[P_n^\gamma \rho_n^{AB}] \\ &= \text{Tr}[P_n^\gamma \sigma_{n,\varepsilon}^{AB}] + \text{Tr}[P_n^\gamma (\rho_n^{AB} - \sigma_{n,\varepsilon}^{AB})] \\ &\leq \text{Tr}[P_n^\gamma (\sigma_{n,\varepsilon}^{AB} - 2^{-n\alpha}I_n^A \otimes \rho_n^B)] \\ &\quad + 2^{-n\alpha} \text{Tr}[P_n^\gamma (I_n^A \otimes \rho_n^B)] + \varepsilon \\ &\leq \text{Tr}[\{\sigma_{n,\varepsilon}^{AB} \geq 2^{-n\alpha}I_n^A \otimes \rho_n^B\}(\sigma_{n,\varepsilon}^{AB} - 2^{-n\alpha}I_n^A \otimes \rho_n^B)] \\ &\quad + 2^{-n(\alpha-\gamma)} + \varepsilon \end{aligned} \quad (82)$$

In the above we have made use of Lemma 1, Lemma 2 and Corollary 1.

Let us choose  $\gamma = \alpha - \delta/2$ , for an arbitrary  $\delta > 0$ , with  $\alpha = \underline{\Upsilon}^\varepsilon(A|B) - \delta/2$ . Then both the first and second terms on the right hand side of (82) goes to zero as  $n \rightarrow \infty$ .

Therefore, for  $n$  large enough and any  $\delta' > 0$ , in the limit  $\varepsilon \rightarrow 0$ , we must have that

$$\text{Tr}(P_n^\gamma \rho_n^{AB}) \leq \delta', \quad (83)$$

which in turn implies that  $\gamma \leq \underline{S}(A|B)$ . Hence, from the choice of the parameters  $\alpha$  and  $\gamma$  it follows that

$$\lim_{\varepsilon \rightarrow 0} \underline{\Upsilon}^\varepsilon(A|B) - \delta \leq \underline{S}(A|B), \quad (84)$$

and since  $\delta$  is arbitrary, we obtain the inequality (77). ■

We next prove the bound

$$\underline{S}(A|B) \leq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\varepsilon(\rho_n^{AB} | \rho_n^B) \quad (85)$$

*Proof of (85):* Let  $\delta > 0$  be arbitrary but fixed. Then by the definition of the inf-spectral conditional entropy rate there exists  $\gamma \in \mathbb{R}$  such that

$$\gamma > \underline{S}(A|B) - \delta \quad (86)$$

and

$$\limsup_{n \rightarrow \infty} \text{Tr}[\{\rho_n^{AB} \geq 2^{-n\gamma}I_n^A \otimes \rho_n^B\} \rho_n^{AB}] = 0. \quad (87)$$

In particular, for any  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,

$$\begin{aligned} & \text{Tr}[\{\rho_n^{AB} > 2^{-n\gamma} \cdot I_n^A \otimes \rho_n^B\} \rho_n^{AB}] \\ & \leq \text{Tr}[\{\rho_n^{AB} \geq 2^{-n\gamma} \cdot I_n^A \otimes \rho_n^B\} \rho_n^{AB}] < \frac{\varepsilon^2}{8}. \end{aligned} \quad (88)$$

Using Lemma 6 we then infer that for all  $n \geq n_0$

$$H_{\min}^\varepsilon(\rho_n^{AB} | \rho_n^B) \geq n\gamma \quad (89)$$

and, hence

$$\liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\varepsilon(\rho_n^{AB} | \rho_n^B) \geq \gamma. \quad (90)$$

Because this holds for any  $\varepsilon > 0$ , we conclude

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\varepsilon(\rho_n^{AB} | \rho_n^B) \geq \gamma > \underline{S}(A|B) - \delta. \quad (91)$$

The assertion (85) then follows because this holds for any  $\delta > 0$ . ■

## CONCLUSIONS

So far, the *information spectrum approach* and the *smooth entropy framework* have been applied within pretty different subfields of information theory [30]. In the quantum regime, spectral entropy rates have mostly been used to characterize information sources, communication channels and entanglement manipulations. In contrast, smooth entropies proved useful in the context of randomness extraction and cryptography. We hope that our result bridges the gap between these two subfields. In fact, for the study of asymptotic settings where the underlying resources are available many times, both the information-spectrum approach and the smooth entropy framework can be used equivalently.

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## APPENDIX

In this Appendix we give the proofs of Proposition 1 and Proposition 2.

### PROOF OF PROPOSITION 1

*Proof* For any  $\alpha = \overline{D}(\rho\|\omega) + \delta$ , with  $\delta > 0$ , implies

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\rho_n] \\ &\geq \lim_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}(\rho_n - e^{n\alpha}\omega_n)] \\ &\geq 0 \end{aligned} \quad (92)$$

giving  $\overline{D}(\rho\|\omega) \geq \overline{D}(\rho\|\omega)$ , as  $\delta$  is arbitrary. For the converse we assume that the inequality is strict, such that  $\overline{D}(\rho\|\omega) = \overline{D}(\rho\|\omega) + 4\delta$  for some  $\delta > 0$ . Then choosing  $\alpha = \overline{D}(\rho\|\omega) + 2\delta$ ,  $\gamma = \overline{D}(\rho\|\omega) + \delta$ , we have from Lemma 1,

$$\begin{aligned} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\rho_n] &\leq \text{Tr}[\{\rho_n \geq e^{n\gamma}\omega_n\}(\rho_n - e^{n\gamma}\omega_n)] \\ &\quad + e^{n\gamma}\text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\omega_n] \\ &\leq \varepsilon_n + e^{-n\delta} \end{aligned} \quad (93)$$

where  $\varepsilon_n = \text{Tr}[\{\rho_n \geq e^{n\gamma}\omega_n\}(\rho_n - e^{n\gamma}\omega_n)]$  and  $\text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\omega_n] \leq e^{-n\alpha}$  holds for any  $\alpha$ . As the right hand side goes to zero asymptotically and since  $\alpha < \overline{D}(\rho\|\omega)$  we have a contradiction. ■

### PROOF OF PROPOSITION 2

*Proof* For any  $\alpha = \underline{D}(\rho\|\omega) - \delta$ , with  $\delta > 0$ , implies

$$\begin{aligned} 1 &\geq \lim_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\rho_n] \\ &\geq \lim_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}(\rho_n - e^{n\alpha}\omega_n)] \\ &= 1 \end{aligned} \quad (94)$$

giving  $\underline{D}(\rho\|\omega) \geq \underline{D}(\rho\|\omega)$ , as  $\delta$  is arbitrary. For the converse we assume that the inequality is strict, such that  $\underline{D}(\rho\|\omega) = \underline{D}(\rho\|\omega) + 4\delta$  for some  $\delta > 0$ . Then choosing

$\alpha = \underline{D}(\rho\|\omega) - \delta$ ,  $\gamma = \underline{D}(\rho\|\omega) - 2\delta$ , we have from Lemma 1,

$$\begin{aligned} 1 &\stackrel{n \rightarrow \infty}{\leftarrow} \text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\rho_n] \\ &\leq \text{Tr}[\{\rho_n \geq e^{n\gamma}\omega_n\}(\rho_n - e^{n\gamma}\omega_n)] \\ &\quad + e^{n\gamma}\text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\omega_n] \\ &\leq \text{Tr}[\{\rho_n \geq e^{n\gamma}\omega_n\}(\rho_n - e^{n\gamma}\omega_n)] + e^{-n\delta} \end{aligned} \quad (95)$$

where  $\text{Tr}[\{\rho_n \geq e^{n\alpha}\omega_n\}\omega_n] \leq e^{-n\alpha}$  holds for any  $\alpha$ . Thus  $\lim_{n \rightarrow \infty} \text{Tr}[\{\rho_n \geq e^{n\gamma}\omega_n\}(\rho_n - e^{n\gamma}\omega_n)] = 1$ , where  $\gamma > \underline{D}(\rho\|\omega)$ , which is a contradiction. ■

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